UDC 534

ANALYTIC EQUIVALENCE OF SECOND-ORDER SYSTEMS FOR ARBITRARY RESONANCE

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We prove a classification theorem which permits us to simplify second-order systems; in particular. to replace series in the right-hand side by polynomials without violating solution properties preservable by analytic transformations in the neighborhood of a singular point. Among such properties are all the topological ones. The most important one of them is stability. Although the question of stability in critical cases (resonances of order $q = 1$, $q = 2$) have been studied in detail in [1-3], an arbitrary resonance in a second-order system yields the simplest nontrivial model the investigation of which helps us to understand the nature of critical cases and its relation with local topological (in particular, analytic) equivalence of systems of differential equations [4, 5].

In contrast to methods of reducing systems to normal form [6], in the present paper we use a group-theoretic approach. The study of the orbits of a group induced in coefficient space by the group of all analytic homeomorphisms of the neighborhood of a singular point, allows us to describe the set of systems of differential equations, obtained from each given system by analytic transformations. Thus, having computed all the transformation invariants we can completely classify the systems being considered. Here we clarify that the normal form is poorly suited for such a classification since it does not contain all representatives of classes of analytically equivalent systems. The case of purely imaginary eigenvalues of the linear part has been considered separately by the author (*).

1. Statement of the problem and the result. We consider a set of second-order systems of differential equations

$$dx_1 / dt = n_1 x_1 + f_1(c, x_1, x_2) \qquad (1.1)$$
$$dx_2 / dt = -n_2 x_2 + f_2(c, x_1, x_2)$$

where n_1, n_2 are relatively prime fixed positive integers; f_1, f_2 are all possible real functions, analytic in a neighborhood of the point $x \equiv \{x_1, x_2\} = 0$, with expansions without linear terms; $c = \{c_1, c_2, \ldots\}$ is an ordered collection of the coefficients of these expansions. The set of all c to which there correspond convergent series, forms an infinite-dimensional space R. A certain point $c \in R$ corresponds to each system (1.1).

Definition 1. The systems $c' \in R$ and $c'' \in R$ are analytically equivalent (locally) if there exists an analytic homeomorphism of a neighborhood of the point

*) Analytic equivalence and stability of second-order systems with 1:1 resonance. Preprint №14, Institute of Problems in Mechanics, Akad. Nauk SSSR, 1972.

$x = 0$ into itself, transforming these systems one into the other.

The problem is to find the necessary and sufficient conditions for the equivalence of systems (1.1) in the sense of Definition 1.

An ordered collection of coefficients of homogeneous forms of fixed degree s, contained in the expansions of functions f_1, f_2, are considered as the coordinates of a point of an Euclidean space R_s^*. We set $R_2 = R_2^*$, $R_s = R_s^* \times R_{s-1}$ (direct product), $s = 3, 4, \ldots$. The order relation for R_s and R on a coincident set of elements is assumed identical. The coefficients of sth-degree polynomials obtained from the expansions of f_1, f_2 by discarding terms of order greater than s, are the coefficients of the point $c_s \in R_s$. If N_s is the total number of these coefficients, then

$$\dim R_s = N_s$$

The space R can be treated as the inductive limit of the sequence R_2, R_3, ...

Let us consider the group G of all analytic transformations of a neighborhood of the point $x = 0$, leaving this point in place and preserving the linear part of system (1.1). Transformations of group G induce the transformation group G': $G' \times R \to R^1$(*), so that every transformation from $G \times G'$ transforms system (1.1) into a system of the same form with a phase vector x' and coefficients c'. It is easily verified that the spaces R_s are invariant relative to transformations from group G', while the collection of transformations from G' not identically acting in R_s, forms a Lie Group G_s'. Here

$$\dim G_s' = N_s + 2$$

Let

$$u = \sum_{k_1, k_2 \in M} a_{k_1 k_2} x_1^{k_1} x_2^{k_2}$$

be an arbitrary polynomial or a formal power series and let N be the set of values of the integral function $k_1 n_1 - k_2 n_2$ for $k_1, k_2 \in M$. If $\nu \in N$, we set

$$u^\nu = \sum_{k_1 n_1 - k_2 n_2 = \nu} a_{k_1 k_2} x_1^{k_1} x_2^{k_2}$$

Then there hold the single-valued expansions

$$u = \sum_\nu u^\nu = \sum_\nu \sum_m u_m{}^\nu, \quad \nu \in N, \quad m = k_1 + k_2$$

With system (1.1) we associate the operator

$$L = (n_1 x_1 + f_1) \frac{\partial}{\partial x_1} + (-n_2 x_2 + f_2) \frac{\partial}{\partial x_2}$$

We consider the formal series $u = u_q + u_{q+1} + \ldots$ (where $q = n_1 + n_2$ is the order of the resonance), satisfying the conditions

$$(Lu)^\nu = 0 \quad \text{for all } \nu \neq 0 \qquad (1.2)$$
$$u^0 = u_q = x_1^{n_2} x_2^{n_1}$$

Then

$$Lu = \sum_{\varkappa=2}^{\infty} (Lu)^\circ_{\varkappa q} \equiv \sum_{\varkappa=2}^{\infty} G_{\varkappa q} \equiv \sum_{\varkappa=2}^{\infty} g_{\varkappa q}(c) (x_1^{n_2} x_2^{n_1})^\varkappa$$

*) The corresponding equations are written out in finite form [7].

Here the parameters $g_{\varkappa q}$ do not depend on the phase vector x, and $\varkappa$ is the exponent of the monomial $u_q = x_1^{n_2} x_2^{n_1}$.

Theorem 1. The collection of manifolds in R

$$\Gamma_h\colon\ g_{2q}(c) = \ldots = g_{hq}^{(c)} = 0, \qquad g_{(h+1)q}(c) \neq 0$$

exhausts, for $h < \infty$, all invariant manifolds of group G'. On each invariant manifold Γ_h the group G' admits of $l + h$ invariants ($l > 1$)

$$I_1(c), \ldots, I_{h+l}(c)$$

For $\varkappa \leqslant h + 1$ the functions $g_{\varkappa q}(c)$ and $I_\varkappa(c)$ depend only on the points $c \in R_{s_0}$, $s_0 = 2qh + 1$. The systems $c' \in R$ and $c'' \in R$ are equivalent if and only if the points c' and c'' lie on one and the same invariant set

$$h' = h'', \qquad I_\varkappa(c') = I_\varkappa(c'') \quad (1 \leqslant \varkappa \leqslant h + l)$$

and belong either to one and the same connection component (points c' and c'' can be connected by a continuous curve lying in the same invariant set on which these points themselves do) or to two different ones provided that there exists a mapping $x_1' \to -x_1$, $x_2' \to x_2$ generating a homeomorphism of these components one into the other.

The following theorem describes the only case when the application of Theorem 1 does not require an actual computation of the invariants.

Theorem 2. System (1.1) is formally equivalent to the system

$$\frac{dx_1}{dt} = n_1 x_1 + \sum_{\mu=2}^{s} f_{1,\mu}$$

$$\frac{dx_2}{dt} = -n_2 x_2 + \sum_{\mu=2}^{s} f_{2,\mu}$$

obtained from system (1.1) by discarding terms of order higher than s in the expansions, if and only if

$$s \geqslant 2qh + 1$$

For one of the simplest cases, $h = 1$, $q = 2$ (pure imaginary eigenvalues of the linear part) the invariants of group G' (there are two) have been computed explicitly. This has allowed us to classify systems admitting of an analytic symmetry group. The special result indicated is contained in the following theorem.

Theorem 3. For $h = 1$, $q = 2$ the set of formally nonequivalent second-order systems is described by the systems (in polar coordinates)

$$\rho^{\bullet} = \rho^3(\sigma_1 + \sigma_2 \varkappa_2 \rho^2) \tag{1.3}$$
$$\varphi^{\bullet} = \sigma_3 + \sigma_4 \varkappa_1 \rho^2$$

when the pair $(\varkappa_1, \varkappa_2)$ of numerical parameters ranges over the whole real plane and the σ_i take the values ± 1 independently of each other.

We note that systems (1.3) are easily integrated and yield 24 topologically different pictures in the space $x \times t$. The proof of Theorems 1 and 2 is carried out in Sect. 3. It is preceded by the formulation of auxiliary propositions (Sect. 2) whose proofs, except for the fundamental Lemma 6, are omitted (they may be reproduced by the scheme given in [7]). Theorem 3 is proven in Sect. 4.

2. Auxiliary propositions. Lemma 0. Conditions (1.2) define the formal series

$$u = x_1^{n_2} x_2^{n_1} + u_{q+1} + \dots \qquad (2.1)$$

uniquely.

Definition 2. A series (2.1) satisfyinf conditions (1.2) is said to be standard.

Definition 3. A formal series v satisfying the equation $Lv = \overset{\circ}{w}_{q(p+1)} + O(q(p+1)+1)$, in which the form $\overset{\circ}{w}_{q(p+1)} \not\equiv 0$ is the lowest term in the expansion of the right-hand side, is called a p-series. The description of all p-series yields the following lemma.

Lemma 1. Let $G_{\varkappa q}$, computed for a standard series, satisfy the conditions

$$G_{2q} = \dots = G_{hq} = 0, \qquad G_{(h+1)q} \neq 0$$

Then:

1) there does not exist a formal series satisfying the equation $Lv = \dot{0}$ in all orders;

2) the set of all p-series coincides with the set of formal series of the form

$$\varphi[u] = au^{p-h+1} + O(q(p-h+1)+1), \quad a \neq 0$$

where u is a standard series, u^{p-h+1} is a power of it;

3) if v is an arbitrary p-series, then

$$Lv = a(p-h+1)\, u_q^{p-h} G_{(h+1)q} + O(q(p+1)+1)$$

Let $\xi = \xi_k + \xi_{k+1} + \dots$, $\eta = \eta_k + \eta_{k+1} + \dots$ be formal power series. The operator series $Z = Z_k + Z_{k+1} + \dots$ $(Z_\mu = \xi_\mu \partial/\partial x_1 + \eta_\mu \partial/\partial x_2)$ is called an operator of order k. We set $Z_\mu^\nu = \xi_\mu^{\nu+n_1} \partial/\partial x_1 + \eta_\mu^{\nu-n_2} \partial/\partial x_2$. As usual, let $[L, Z]$ be a commutator.

Lemma 2. If an operator Z of order μ satisfies the equation $[L, Z] = 0$ to within terms of order $m > \mu$, then $Z_\mu = Z_\mu^\circ$, necessarily,

$$Z_\mu = \begin{cases} 0, & \mu \neq kq+1 \\ u_q^{(\mu-1)/q}(\alpha_\mu X_1 + \beta_\mu L_1), & \mu = kq+1 \end{cases}$$

where α_μ, β_μ are constants,

$$X_1 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \qquad L_1 = n_1 x_1 \frac{\partial}{\partial x_1} - n_2 x_2 \frac{\partial}{\partial x_2}$$

Lemma 3. Let the operator $Z = Z_\mu + Z_{\mu+1} + \dots$ satisfy the conditions

$$[L, Z]^\nu = 0 \text{ for all } \nu \neq 0, \qquad Z^\circ = U^\circ \qquad (2.2)$$

where the operator U° is preassigned. Conditions (2.2) define operator Z uniquely. The identity

$$[L, Z] = [L, Z]^\circ = \sum_{\varkappa=p}^{\infty} [L, Z]^\circ_{\varkappa q+1}, \qquad [L, Z]^\circ_{pq+1} \neq 0 \qquad (2.3)$$

is valid for the operator Z defined by conditions (2.2).

Definition 3. The operator $Z_{(\mu)} = Z_{\mu q+1} + Z_{\mu q+2} + \dots$, satisfying identity (2.3) is called a p-operator. If the operator U° is chosen so that the number p is maximal, operator Z is called maximal.

According to Lemma 2, maximal p-operators necessarily form the linear hull of the set of independent operators of the form

$$X_{(\mu)} = u_q{}^{\mu} X_1 + X_{\mu q+2} + \ldots, \qquad Y_{(\mu)} = u_q{}^{\mu} L_1 + Y_{\mu q+2} + \ldots \tag{2.4}$$

The immediate problem is to compute for them the positive integers p and $\tau = q\,(p - \mu)$.

Lemma 4. Let the operator $X \equiv X_{(0)} = X_1 + X_2 + \ldots$ be maximal. We have

$$[L, X] = \overset{\circ}{P}_{qm+1} + \overset{\circ}{P}_{q(m+1)+1} + \ldots, \qquad \overset{\circ}{P}_{qk+1} \equiv [L, X]^{\circ}_{qk+1}$$

The inequality $m \leqslant h$ is valid.

Lemma 5. The equality $[L, Z_{(\mu)}] = \overset{*\circ}{P}_{pq+1} + \overset{*\circ}{P}_{(p+1)q+1} + \ldots$, in which

$$\overset{**}{P}_{pq+1} = \begin{cases} a_{\mu} u_q{}^{\mu} \overset{\circ}{P}_{mq+1}, & m < h, \quad a_{\mu} \neq 0 \\ a_{\mu} u_q{}^{\mu} \overset{\circ}{P}_{mq+1} + \mu u_q^{\mu-1} G_{(h+1)q}\,(a_{\mu} X_1 + b_{\mu} L_1), & m = h \end{cases}$$

is valid, independently of the choice of $\overset{\circ}{Z}_{(\mu)}$, for the maximal operator $Z_{(\mu)} = u_q{}^{\mu} (a_{\mu} X_1 + b_{\mu} L_1) + \ldots\ (b_{\mu}{}^2 + a_{\mu}{}^2 \neq 0)$.

Lemma 6. (1) When $a_{\mu} \neq 0$ the maximal operators $Z_{(\mu)}$ are contained among the $X_{(\mu)}$. For them

$$p = \mu + m, \qquad \tau = qm$$

2) When $a_{\mu} = 0$, $\mu \neq m$ the maximal operators $Z_{(\mu)}$ are contained among the operators $Y_{(\mu)}$. For them

$$p = \mu + 2h - m, \qquad \tau = q\,(2h - m)$$

3) For $\mu = m$ and for finite h there exists a unique operator $Z_{(m)} \equiv Y_{(m)}$ satisfying the equation $[L, Z] = 0$ in all orders. For it $\tau = \infty$.

Proof. By virtue of Lemmas 4 and 5 the proof of Lemma 6 splits up into the cases: (1) $m = h$; (2) $m < h$, $a_{\mu} \neq 0$; (3) $m < h$, $a_{\mu} = 0$.

1) Let $m = h$. Then $[L, X] = \overset{\circ}{P}_{qh+1} + \overset{\circ}{P}_{q(h+1)} + \ldots$. Let us ascertain the structure of operator $\overset{\circ}{P}_{qh+1}$. For a standard series u,

$$L\,(Xu) = [L, X]\, u + XLu = \overset{\circ}{P}_{qh+1} u_q + q\,(h+1)\, G_{(h+1)q} + \ldots$$

$$\overset{\circ}{P}_{ph+1} = u_q{}^{h}\,(\alpha X_1 + \beta L_1)$$

Hence

$$L\,(Xu) = q\,(\alpha + (h+1)\, g_{(h+1)q})\, u_q^{h+1} + \ldots$$

Let us show that $\alpha + (h+1)\, g_{(h+1)q} \neq 0$. Assuming the contrary, we find $L\,(Xu) = w_{q(h+1)+1} + \ldots \equiv w$. We define the series $\omega = \omega_{q(h+1)+1} + \ldots$ by the relation $(L\omega)^{\nu} = w^{\nu}$ solvable for all $\nu \neq 0$. Then,

$$Lv = w - L\omega = w^{\circ} - (L\omega)^{\circ} = \overset{\prime\circ}{w}_{q(h+k)} + \ldots \qquad (k > 1)$$

i.e. $v \equiv Xu - \omega = qu_q + \ldots$ is an $(h+k)$-series, which is impossible when $k > 1$ (Lemma 1). Hence, using Lemma 1 repeatedly, we obtain

$$\alpha = -h g_{(h+1)q}, \quad \overset{\circ}{P}_{qh+1} = u_q{}^{h}\,(-h g_{(h+1)q} X_1 + \beta L_1)$$

By the formula in Lemma 5 we find

$$\overset{*\circ}{P}_{qp+1} = (\mu - h)\, a_{\mu} g_{(h+1)q} u_q^{h+\mu} X_1 + b_{\mu}{}' u_q^{h+\mu} L_1$$

$$b_{\mu}{}' = \beta a_{\mu} + \mu b_{\mu} g_{(h+1)q}$$

Thus, for $m = h$,

$$[L, Z_{(\mu)}] = u_q^{\mu+h}\, [(\mu - h)\, a_{\mu} g_{(h+1)q} X_1 + b_{\mu}{}' L_1] \tag{2.5}$$

and this result does not depend on the choice of the operators $\overset{\circ}{Z}_{q(\mu+1)+1}, \overset{\circ}{Z}_{q(\mu+2)+1}, \ldots$. In other words, when $m=h$, $\mu\neq h$ (if $a_\mu=0$, then $b_\mu\neq 0$) the operator $Z_{(\mu)}$ is maximal and for it

$$p=\mu+h, \qquad \tau=qh$$

When $\mu=h$ we take $b_\mu=-\beta a_\mu/\mu g_{(h+1)q}$. Then

$$b'_\mu=0, \qquad [L, Z_{(h)}]=\widetilde{P}^{\circ}_{q(2h+1)+1}+\ldots$$

By formula (2.5),

$$[L, Z_{(h+1)}]=u_q^{2h+1}[a_{h+1}g_{(h+1)q}X_1+b'_{h+1}L_1]+\ldots$$
$$b'_{h+1}=a_{h+1}\beta+(h+1)b_{h+1}g_{(h+1)q}$$

Since $P^{*\circ}_{q(2h+1)+1}=u_q^{2h+1}(\alpha_1X_1+\beta_1L_1)$, having taken

$$a_{h+1}=\frac{\alpha_1}{g_{(h+1)q}}, \qquad b_{h+1}=\frac{\beta_1-a_{h+1}\beta}{(h+1)g_{(h+1)}}$$

we obtain

$$[L, Z_{(h)}-Z_{(h+1)}]=\widetilde{P}^{\circ}_{g(2h+2)+1}+\ldots$$

Acting analogously, i.e. choosing in the operators $Z_{(h+k)}=u_q^{h+k}(a_{h+k}X_1+b_{h+k}L_1)+\ldots$ the parameters a_{h+k}, b_{h+k} from the formulas

$$a_{h+k}=\frac{\alpha_k}{g_{(h+1)q}}, \qquad b_{h+k}=\frac{\beta_k-a_{h+k}\beta}{(h+k)g_{(h+k)q}}$$

we construct the unique operator $Z=Z_{(h)}-Z_{(h+1)}-Z_{(h+2)}-\ldots$, satisfying the equation $[L, Z]=0$ in all orders.

2) Let $m<h$, $a_\mu\neq 0$. According to Lemma 5

$$[L, Z_{(\mu)}]=a_\mu u_q^\mu \overset{\circ}{P}_{qm+1}+\ldots$$

Since this result is independent of the choice of operator $Z^{\bullet}_{(\mu)}$, the operator $Z_{(\mu)}$ is maximal and

$$p=\mu+m, \qquad \tau=qm$$

Without loss of generality $b_\mu=0$, so that $Z_{(\mu)}$ is found among the operators $X_{(\mu)}$.

3) Now let $m<h$, $a_\mu=0$. By means of the constructions already used earlier we see that $\overset{\circ}{P}_{qm+1}u_q=0$, whence

$$\overset{\circ}{P}_{qm+1}=\beta_m u_q{}^m L_1, \qquad \beta_m\neq 0 \tag{2.6}$$

If $Y_{(\mu)}=u_q{}^\mu L_1+Y_{q\mu+2}+\ldots$ is a maximal operator, this signifies that the values of $\overset{\circ}{Y}_{q(\mu+\gamma)+1}=u_q^{\mu+\gamma}(a^{(\gamma)}_{\mu+\gamma}X_1+b^{(\gamma)}_{\mu+\gamma}L_1)$ are chosen for $\gamma\geqslant 1$ in such a way that the number p in the relation

$$[L, Y_{(\mu)}]=\overset{\circ}{Q}_{qp+1}+\overset{\circ}{Q}_{q(p+1)+1}+\ldots$$

is maximal. It is not difficult to prove that from the maximality of p follows

$$\overset{\circ}{Q}_{qp+1}u_q\neq 0 \tag{2.7}$$

Further, let $a^{(\mu)}_{\mu+\gamma}=0$ for $\gamma<\gamma_0$ and $a^{(\gamma_0)}_{\mu+\gamma_0}\neq 0$. For $1\leqslant\gamma<\gamma_0$ we define inductively the operators

$$Y^{(0)}_{(\mu)}\equiv Y_{(\mu)}, \qquad Y^{(\gamma)}_{(\mu)}=Y^{(\gamma-1)}_{(\mu)}-u^{\mu+\gamma}b^{(\mu)}_{\mu+\gamma}L \tag{2.8}$$

By induction on γ and by direct verification for $\gamma=1$ we see that the formulas

$$[L, Y^{(\gamma)}_{(\mu)}]=-\mu u_q^{\mu-1}G_{(h+1)q}L_1+\ldots$$
$$Y^{(\gamma)}_{(\mu)}=b^{(\mu)}_{\mu+\gamma}u_q^{\mu+\gamma}L_1+Y^{(\gamma)}_{q(\mu+\gamma)+2}+\ldots \qquad \gamma\leqslant h-1$$

are true for $\gamma = 1, \ldots, \gamma_0 - 1$. We define

$$Y_{(\mu)}^{(\gamma_0)} = Y_{(\mu)}^{(\gamma_0-1)} - u^{\mu+\gamma_0-1} b_{\mu+\gamma_0-1}^{(\mu)} L = Y_{q(\mu+\gamma_0-1)+2}^{(\gamma_0)} + \ldots$$

We obtain

$$[L, Y_{(\mu)}^{(\gamma_0)}] = -\mu u_q^{\mu-1} G_{(h+1)q} L_1 + \ldots$$

Since $\gamma_0 \leqslant h$, from this we have

$$Y_{q(\mu+\gamma_0-1)+2}^{(\gamma_0)} = \ldots = Y_{q(\mu+\gamma_0)}^{(\gamma_0)} = 0$$

$$Y_{q(\mu+\gamma_0)+1}^{(\gamma_0)} = \left(Y_{q(\mu+\gamma_0)+1}^{(\gamma_0)}\right)^{\circ} = u_q^{\mu+\gamma_0} \left(a_{\mu+\gamma_0}^{(\mu)} X_1 + b_{\mu+\gamma_0}^{(\mu)} L_1\right)$$

Further,

$$Y_{(\mu)}^{(\gamma_0+1)} = Y_{(\mu)}^{(\gamma_0)} - u^{\mu+\gamma_0} \left(a_{\mu+\gamma_0}^{(\mu)} X + b_{\mu+\gamma_0}^{(\mu)} L\right) = Y_{q(\mu+\gamma_0)+2}^{(\gamma_0+1)} + \ldots$$
$$[L, Y_{(\mu)}^{(\gamma_0+1)}] = -\mu u_q^{\mu-1} G_{(h+1)q} L_1 - a_{\mu+\gamma_0}^{(\mu)} u_q^{\mu+\gamma_0} P_{qm+1}^{\circ} + \ldots \tag{2.9}$$

The equality

$$\Omega^{\circ} \equiv \mu u_q^{\mu-1} G_{(h+1)q} L_1 + a_{\mu+\gamma_0}^{(\mu)} u_q^{\mu+\gamma_0} P_{qm+1}^{\circ} = 0$$

is necessarily fulfilled, from which follows

$$\gamma_0 = h - m$$

since any of the three assumptions

1) $\gamma_0 > h - m$, 2) $\gamma_0 < h - m$, 3) $\gamma_0 = h - m$, $\Omega^{\circ} \neq 0$

leads to a contradiction (this is proved by means of constructions of a single type).

Thus,

$$a_{\mu+1}^{(\mu)} = \ldots = a_{\mu+h-m-1}^{(\mu)} = 0, \qquad a_{\mu+h-m}^{(\mu)} \neq 0 \tag{2.10}$$

Using the recurrence relations (2.8) and equalities (2.9) and (2.10), we find

$$Y_{(\mu)} = \varphi L + a_{\mu+h-m}^{(\mu)} u^{\mu+h-m} X + Y_{q(\mu+\gamma_0)+2}^{(\gamma_0+1)} + \ldots$$
$$\varphi = u^{\mu} + b_{\mu+1}^{(\mu)} u^{\mu+1} + \ldots + b_{\mu+h-m}^{(\mu)} u^{\mu+h-m}$$

where φ is a polynomial of degree $\mu + h - m$ relative to a standard series. We consider the series $w \equiv Y_{(\mu)} u$. We have

$$w = q a_{\mu+h-m}^{(\mu)} u_q^{\mu+h-m+1} + \ldots$$

We denote $G = G_{(h+1)q} + G_{(h+2)q} + \ldots$ and we introduce the series

$$w' = w - \varphi G = q a_{\mu+h-m}^{(\mu)} u_q^{\mu+h-m+1} + \ldots$$

A simple calculation yields

$$Lw' = Q_{qp+1}^{\circ} u_q + (h+1)\, q a_{\mu+h-m}^{(\mu)} g_{(h+1)q} u_q^{\mu+2h-m+1} + \ldots \tag{2.11}$$

Let $q(p+1) < q(\mu + 2h - m + 1)$. Then $Lw' = Q_{qp+1}^{\circ} u_q + \ldots$ and w' is a p-series. According to Lemma 1, $\mu + h - m + 1 = p - h + 1$ and, consequently, $q(p+1) = q(\mu + 2h - m + 1)$, in spite of the assumption. If $q(p+1) > q(\mu + 2h - m + 1)$, then

$$Lw' = q(h+1)\, a_{\mu+h-m}^{(\mu)} u_q^{h-m+\mu} G_{(h+1)q} + \ldots$$

and w' is a $(\mu + 2h - m)$-series. But by Lemma 1 we should have

$$Lw' = (\mu + h - m + 1)\, q a_{\mu+h-m}^{(\mu)} u_q^{\mu+h-m} G_{(h+1)q} + \ldots$$

which is impossible when $\mu \neq m$.

Thus, if $\mu \neq m$, the equality $q(p+1) = q(\mu + 2h - m + 1)$ is fulfilled, whence $p = \mu + 2h - m$. Here we obtain

$$\overset{\circ}{Q}_{qp+1} u_q = (\mu - m)\, q a^{(\mu)}_{\mu+h-m} u_q^{\mu+h-m} G_{(h+1)q} + \dots$$

so that

$$[L, Y_{(\mu)}] = (\mu - m)\, a^{(\mu)}_{\mu+h-m} g_{(h+1)q} u_q^{\mu+2h-m} X_1 \dot{+} A_\mu u_q^{\mu+2h-m} L_1 + \dots \qquad (2.12)$$

Thus, the operator $X_{(\mu)}$ is maximal for $\mu \neq m$ and

$$p = \mu + 2h - m, \qquad \tau = q(2h - m)$$

Now let $\mu = m$. Then $w' = q a_h^{(m)} u_q^{h+1} + \dots$ and by Lemma 1

$$Lw' = q(h+1)\, a_h^{(m)} g_{(h+1)q} u_q^{2h+1} + \dots \qquad (2.13)$$

Hence it follows that $p > 2h$. Indeed, the assumption $p < 2h$, ensuing from $q(p+1) < q(\mu + 2h - m + 1)$ with $\mu = m$, leads to a contradiction as we have shown. From $p = 2h$ follows $\overset{\circ}{Q}_{qp+1} u_q = 0$ (by means of comparing formulas (2.11) and (2,13)), which contradicts (2.7). Thus, $p > 2h$ in the relation

$$[L, Y_{(m)}] = \overset{\circ}{Q}_{qp+1} + \overset{\circ}{Q}_{q(p+1)+1} + \dots$$

Let us show that in fact there is no finite, p whatsoever that can be maximal. Indeed, let

$$\overset{\circ}{Q}_{qp+1} = u_q^p (C X_1 + D L_1), \qquad C^2 + D^2 \neq 0$$

We consider the maximal operators $Y_{(p+m-2h)}$, $X_{(p-m)}$. For them

$$[L, Y_{(p+m-2h)}] = a^{(p+m-2h)}_{p-h} g_{(h+1)q} (p - 2h)\, u_q^p X_1 + A_{p+m-2h} u_q^p L_1 + \dots$$

$$(a^{(p+m-2h)}_{p-h} \neq 0)$$

$$[L, X_{(p-m)}] = a_{p-m} \beta_m u_q{}^p L_1 + \dots$$

$$(a_{p-m} \beta_m \neq 0)$$

We determine the numbers α and β by the formulas

$$\alpha a^{(p+m-2h)}_{p-h} g_{(h+1)q} (p - 2h) = C$$

$$\alpha A_{p+m-2h} + \beta \beta_m a_{p-m} = D$$

This is possible since the determinant of this system

$$\Delta = (p - 2h)\, \beta_m a_{p-m} a^{(p+m-2h)}_{p-h} g_{(h+1)q} \neq 0$$

The operator $Y = Y_{(m)} - \alpha Y_{(p+m-2h)} - \beta X_{(p-m)}$ satisfies the equation

$$[L, Y] = \overset{\prime\circ}{Q}_{qp_1+1} + \overset{\prime\circ}{Q}_{q(p_1+1)+1} + \dots$$

in which $p_1 \geqslant p + 1$. Thus, for $\mu = m$ we can construct an operator $Y_{(m)}$ satisfying the equation $[L, Y_{(m)}] = 0$ in all orders. The operator $Y_{(m)}$ is unique. Indeed, if each of the two operators $Y'_{(m)} = u_q^m L_1 + Y'_{qm+2} + \dots$ and $Y''_{(m)} = u_q^m L_1 + Y''_{qm+2} + \dots$ were to satisfy the equation $[L, Y_{(m)}] = 0$, we would obtain

$$[L, Y'_{(m)} - Y''_{(m)}] = 0, \quad Y'_{(m)} - Y''_{(m)} = u_q^{m+1} (a X_1 + b L_1) + \dots$$

which is impossible because an expansion of an operator satisfying the equation $[L, Y] = 0$ in all orders should start with an operator of order $qm + 1$. Lemma 6 is proven.

3. Proof of Theorems 1 and 2. The proof is based on an enumeration of the invariant sets of group $\mathbf{G}'$. It is convenient to pass from the groups $\mathbf{G}'$ and $\mathbf{G} \times \mathbf{G}'$ to their algebras $\mathbf{L}$ and $\mathbf{L}^*$ of the operators

$$Z = \xi \frac{\partial}{\partial x_1} + \eta \frac{\partial}{\partial x_2} \in \mathbf{L}, \qquad Z^* = Z + \sum_i \zeta_i (c) \frac{\partial}{\partial c_i} \in \mathbf{L}^* \tag{3.1}$$

The condition for the invariance of system (1.1) relative to the transformations from group $\mathbf{G} \times \mathbf{G}'$ yields $[L, Z^*] = 0$ or equivalently

$$[L, Z] = \sum_i \left(\zeta_i (c) \frac{\partial f_1}{\partial c_i} \frac{\partial}{\partial x_1} + \zeta_i (c) \frac{\partial f_2}{\partial c_i} \frac{\partial}{\partial x_2} \right) \tag{3.2}$$

Equality (3.2) must be fulfilled identically in x_1, x_2 and can serve to compute the elements $\zeta_i^j (c)$ of the vector matrix (ζ_i^j) of the algebra corresponding to group $\mathbf{G}'$ (in the natural basis). From equality (3.2) we see at once that if Z is an arbitrary operator of order μ, then the expansion of the right-hand side of (3.2) with respect to x_1, x_2 starts, generally speaking, with terms of order μ Hence $(\zeta_i^j (c) \equiv 0$ for all i which correspond to coefficients of powers of $f_1\ f_2$ less than μ. Hence we have a block-triangular structure of the matrix (ζ_i^j) (the zeros are in the lower left corner (*)). If operator Z is maximal, then, in addition, it makes zeros out of all elements of its own row, belonging to $\tau = q\ (p - \mu)$ nonzero blocks. Here, this number cannot be increased by any linear combination of operator Z with higher-order operators.

Let us consider the space R_s. Operators $Z \in \mathbf{L}$, corresponding to nonidentity (a priori) transformations of space R_s, form a certain set $\mathbf{L}_s$ (which is not a Lie algebra). Let r_s be the maximum number of operators $Z_{(\mu)} \in \mathbf{L}_s$ such that

$$[L, Z_{(\mu)}] = O^* (qp + 1), \qquad qp + 1 > s \tag{3.3}$$

where $O^* (qp + 1)$ is an operator of order $qp + 1$. From $Z_{(\mu)} \in \mathbf{L}_s$ follows

$$q\mu + 1 \leqslant s \tag{3.4}$$

A comparison of formulas (3.2) and (3.3) shows that in the vector matrix (ζ_i^j) corresponding to group $\mathbf{G}_s'$ we can form exactly r_s-rows consisting of zeros. This signifies that group $\mathbf{G}_s'$ admits of precisely $\rho_s = r_s - 2$ functionally independent invariant sets (**). We note that no role is played by the formality of the majority of the expansions (for finite h) examined in this paper; for all maximal operators, besides $Y_{(m)}$, the number τ is in fact determined by only a finite number of terms of the expansion. The analyticity of operator $Y_{(m)}$ either does not hold at all (then $l > 1$) or follows from the assumption on the existence of an analytic symmetry group for the original equations.

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*) See [8] for details of the structure of the matrix (ζ_i^j).

Editor's Note: There is no reference [8] in the original Russian paper. Correction of this obvious misprint is impossible.

**) Here, by an invariant set we mean and invariant manifold or a one-dimensional continuum of hypersurfaces specified by an invariant.

By $\alpha(X)$ we denote the number of maximal operators $X_{(\mu)}$ satisfying conditions (3.3) and (3.4); by $\beta_1(Y)$, the number of maximal operators $Y_{(\mu)}$ satisfying condition (3.4); by $\beta_2(Y)$, the number of maximal operators $Y_{(\mu)}$ not satisfying condition (3.3). Then the number of invariant sets can be computed by the formula

$$\rho_s = \alpha(X) + \beta_1(Y) - \beta_2(Y) - 2 \tag{3.5}$$

Let us compute the number ρ_s for $s = 2qh$. The quantity $\alpha(X)$ equals the number of integral solutions (relative to μ) of the inequalities

$$q\mu + 1 \leqslant 2qh < q\mu + 1 + \tau \qquad (\tau = qm)$$

whence $\alpha(X) = m$. The quantity $\beta_1(Y)$ equals the number of integral solutions of the inequalities

$$q\mu + 1 \leqslant 2qh, \qquad \mu \geqslant 0$$

whence $\beta_1(Y) = 2h$. The quantity $\beta_2(Y)$ equals the number of integral solutions of the inequality

$$qp + 1 \leqslant 2qh, \qquad \mu \neq 0 \qquad (p = \mu + 2h - m)$$

(the value $\mu = 0$ is excluded because $Y_{(0)} \equiv L$ satisfies condition (3.3)). Hence $\beta_2(Y) = m - 1$. By formula (3.5)

$$\rho_s = 2h - 1 \qquad (s = 2qh) \tag{3.6}$$

Now let $s = 2qh + qk + k_1$, $k \geqslant 0$, $0 \leqslant k_1 < q$, $k^2 + k_1^2 \neq 0$. The quantity $\alpha(X)$ equals the number of integral solutions of the inequalities

$$q\mu + 1 \leqslant 2qh + qk + k_1 < q\mu + 1 + \tau \quad (\tau = qm)$$

whence $\alpha(X) = m$. The quantity $\beta_1(Y)$ equals the number of integral solutions of the inequalities

$$q\mu + 1 \leqslant 2qh + qk + k_1, \qquad \mu \geqslant 0$$

Hence

$$\beta_1(Y) = \begin{cases} 2h + k, & k_1 = 0 \\ 2h + k + 1, & k_1 > 0 \end{cases}$$

The quantity $\beta_2(Y)$ equals the number of integral of the inequality

$$q\mu + 1 + \tau \leqslant 2qh + qk + k_1, \qquad \mu \neq 0,\ m$$
$$(\tau = q(2h - m))$$

(the values $\mu = 0$, m are excluded since the operators $Y_{(0)}$, $Y_{(m)}$ satisfy condition (3.3)); we obtain

$$\beta_2(Y) = \begin{cases} m + k - 2, & k_1 = 0 \\ m + k - 1, & k_1 > 0 \end{cases}$$

Thus, independently of m, k, k_1

$$\rho_s = 2h \qquad (s \geqslant 2qh + 1) \tag{3.7}$$

By comparing formulas (3.6) and (3.7) we see that beginning with the number $s_0 = 2qh + 1$ the groups G_s' $(s \geqslant s_0)$ acting in R_s as transformation groups, have one and the same number $(2h)$ of invariant sets.

The invariant sets of group G_{s_0}' depend, obviously, only on the points of space R_{s_0}. Moreover, each of them remaining invariant for all groups G_s' $(s > s_0)$, is also invariant for the group G' (this follows from the invariance of the subspace R_{s_0} relative to the action of group G'). Besides these invariant sets the group G' can have only those which are consequences of the convergence requirement of the transformations.

From Lemma 6 it follows that the numbers τ for maximal operators vary together with h. However, the variations of the numbers τ is accompanied by the variation of the rank of the group matrices (ζ_i^j). Therefore, the manifolds

$$g_{2h} = 0;\ g_{2q} = g_{3q} = 0; \ldots;\ g_{2q} = \ldots = g_{hq} = 0 \tag{3.8}$$

are invariant manifolds of group G', and a further lowering of the rank of the mapping $G' \times R \to R$ is possible only for $g_{(h+1)q} = 0$. The number of invariant manifolds (3.8) equals $h-1$ and the number of finite-dimensional invariants equals $h+1$. It is clear that two systems of equations of form (1.1) are equivalent if and only if the points $c' \in R$, $c'' \in R$ corresponding to them belong to one and the same orbit of group G'. For this they must lie on one and the same invariant set of group G', whence

$$h' = h'',\ J_1(c') = J_1(c''), \ldots,\ J_{h+l}(c') = J_{h+l}(c'')$$

($J_\varkappa(c)$ are the invariants of G'). Moreover, the points c' and c'' must lie either in one connection component or in connection components which are congruent relative to reflection. In the latter case the transformation taking c' into c'' (or c'' into c'), is not an element of a continuous one-parameter transformation. Theorem 1 is proved.

If as the simplest representations of systems (1.1) we take those which are obtained from system (1.1) by a simple discarding of all expansion terms beginning with some power $s+1$, then all the hypotheses of Theorem 1 are fulfilled for formal thansformations when $s \geqslant 2qh+1$. This proves Theorem 2.

4. Proof of Theorem 3. For $h = 1$ and $q = 2$ (a pair of pure imaginary roots) the number of invariants equals two. For the standard series $u = z\bar{z} + u_3 + \ldots$ and the operator $X_{(0)} = X_1 + X_2 + \ldots$ we have the formulas

$$Lu = g_4(c)\,u_2^2 + g_6(c)\,u_2^3 + \ldots, \qquad u_2 = z\bar{z}$$
$$[L, X_{(0)}] = P_3^0 + P_5^0 + \ldots = u_2(-g_4(c)\,X_1 + \beta_1 L_1) + P_5^0 + \ldots$$

We can check that the functions

$$J_1(c) = \frac{\beta_1(c)}{g_4(c)} \quad J_2(c) = \frac{g_6(c)\,u_2^3 + 2g_4(c)\,u_2\psi_1 + g_4(c)\,\psi_2}{g_4^2(c)\,u_2^3}$$

$$\left(\psi_1 = \sum_{\mu=-3}^{3} \frac{1}{\mu^2} L_2^{-\mu} L_2^{\mu} u_2,\quad \psi_2 = \sum_{\mu=-3}^{3} \frac{1}{\mu^2} L_2^{\mu} u_2 \cdot L_2^{-\mu} u_2\right)$$

are invariants of group G' (the verification is conducted in terms of operators). The parameters $g_4(c)$ and $g_6(c)$ have the forms

$$g_4(c) = \frac{1}{u_2^2} L_3^0 u_2 + \ldots, \qquad g_6(c) = \frac{1}{u_2^3} L_5^0 u_2 + \ldots$$

where the terms not written out do not depend on L_3 and L_5, respectively. Therefore, the system of equalities $J_1(c) = \varkappa_1$, $J_2(c) = \varkappa_2$ is single-valued and continuously solvable with respect to the coefficients of the third and the fifth powers in the expansions of the right-hand sides in Eqs. (1.1). Consequently, these equations describe a simply-connected (smooth) set in R_5. By virtue of the single-valuedness and of the continuous solvability of the equation $g_4(c) = 0$ with respect to one of the coefficients of operator L_3°, the set $g_4(c) \neq 0$ consists of two simply-connected parts: $g_4(c) > 0$ and $g_4(c) < 0$.

Thus, all possible orbits of group G' yield two types of relations

$$J_1(c) = \varkappa_1, \quad J_2(c) = \varkappa_2, \quad g_4(c) > 0; \quad J_1(c) = \varkappa_1, \quad J_2(c) = \varkappa_2, \quad g_4(c) < 0$$

Having chosen as the simplest form of Eqs. (1.1) the normal form and having computed the invariants J_1, J_2 for it and allowed for the sign of g_4, we are convinced in the validity of Theorem 3 after passing to polar coordinates.

Note. The author acknowledges A.D.Briuno for having drawn his attention to the important examples from [6]. After analyzing them the author refined, in the galley proofs, a number of formulations connected with the limit passage from R_s to R. The author considers it important to note that the difficulty of the limit passage is surmounted in a unified manner by using the group-theoretic approach developed here. It was shown, for example, that a group G' acting in the coefficient space of the system $\dot{x} = x^2$, $\dot{y} = y + b_0 x + \ldots + b_k x^{k+1} + \ldots$ is intransitive and admits of a single (limit) invariant $I = b_0 + \ldots + b_k \mid k! + \ldots$ arising from the requirement of convergence of the transformations. The systems indicated lend themselves to a complete classification: only those ones are analytically equivalent for which the numerical values of invariant I coincide. When $I = 0$ the system is equivalent to its own normal form, which agrees with the Briot-Bouquet formula (see [6], p.125). These equations admit of an analytic symmetry group only when $I = 0$.

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